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Numerical Computation of 2D Sommerfeld Integrals— A Novel Asymptotic Extraction Technique

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The accurate and efficient computation of the elements in the impedance matrix is a crucial step in the application of Galerkin's method to the analysis of planar structures. As was demonstrated in a previous paper, it is possible to decompose the angular integral, in the polar representation for the 2D Sommerfeld integrals, in terms of incomplete Lipschitz–Hankel integrals (ILHIs) when piecewise sinusoidal basis functions are employed. Since Bessel series expansions can be used to compute these ILHIs, a numerical integration of the inner angular integral is not required. This technique provides an efficient method for the computation of the inner angular integral; however, the outer semi-infinite integral still converges very slowly when a real axis integration is applied. Therefore, it is very difficult to compute the impedance elements accurately and efficiently. In this paper, it is shown that this problem can be overcome by using the ILHI representation for the angular integral to develop a novel asymptotic extraction technique for the outer semi-infinite integral. The usefulness of this asymptotic extraction technique is demonstrated by applying it to the analysis of a printed strip dipole antenna in a layered medium.

1. INTRODUCTION

Monolithic microwave and millimeter-wave integrated circuits (MMICs) comprise an emerging technology with a wide variety of applications in the field of computers, remote sensing, signal processing, and communications. Due to the planar geometries that are inherent in most passive MMIC structures, the spectral domain technique can often be used for their analysis.

The spectral domain technique is a very general method which can be used to construct an "exact" integral equation for planar geometries [1]. The structure can have an arbitrary number of layers, where each layer is characterized by a thickness, permeability, and a complex permittivity, which enables losses to be accounted for. It is a very powerful technique since it accounts for radiation and mutual coupling effects. These effects are very important for the accurate analysis of densely packed MMIC circuits.

After the integral equation has been formulated for the problem, the next step involves solving for the unknown current distribution on the structure. An approximate solution can be obtained by employing the method of moments (MOM). This method involves approximating the integral equation for the structure by a set of linear equations which can then be solved using standard matrix techniques. However, before the system of linear equations can be solved, the elements in the impedance matrix, which can be represented in the form of two-dimensional (2D) Sommerfeld integrals, must be computed. It is important to develop efficient techniques for the calculation of the elements in the impedance matrix since this task usually requires a large percentage of the total computation time in a MOM problem.

As was demonstrated in [2], application of Galerkin's method to printed circuits in planar geometries involves filling an impedance matrix whose elements have the general form [2, (7)], where the angular integral is given by [2, (8)]. This is the spectral domain representation for the impedance elements. As is noted in [2], the impedance elements can also be represented in the spatial domain. The spatial domain representation involves a 4D spatial integration of the Green's function for the problem.

The integral in [2, (7)] will be referred to as a 2D Sommerfeld integral since the angular integral cannot be expressed in terms of Bessel functions like it can in the case

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of the 1D Sommerfeld integrals found in the Green's function [3–6].

Reference to [2, (7), (8)] shows that efficient techniques must be developed for both the inner angular integral and the outer semi-infinite integral. The problem concerning the accurate and efficient computation of the angular integral is addressed in [2]. In that paper, it was shown that the angular integral can be decomposed into a finite number of incomplete Lipschitz–Hankel integrals (ILHIs) which have the general form [2, (21)]. Therefore, the angular integral, in a 2D Sommerfeld integral, can be represented in terms of special functions just like in the case of the 1D Sommerfeld integrals (i.e., the spatial domain Green's function). However, the ILHIs are more difficult to compute than the Bessel functions that were obtained in the case of the 1D Sommerfeld integrals.

It turns out that the decomposition of the angular integral in terms of ILHIs not only yields an efficient technique for the computation of the inner integral, but it can also be used as the starting point for the derivation of a novel asymptotic extraction technique (AET) for the outer semi-infinite integral. The general idea is to use the series expansions for the ILHIs, that were derived in [7] and used in [2], to find an asymptotic expansion for the integrand of [2, (7)] which holds for large values of the spectral variable λ, and which when integrated from some lower limit L to infinity can be efficiently calculated. If this can be accomplished, then numerical integration will only be required over the range from 0 to L, thereby significantly improving the efficiency of the algorithm. A novel AET will be derived in this paper using the procedure which is described above.

It is very important from a computational point of view to find some method for handling the asymptotic portion of the semi-infinite integral since the integrand only decays algebraically for large values of the spectral variable λ when a real-axis integration is used. Using an AET not only improves the computational efficiency, but it also improves the accuracy of the final result.

AETs are used in [3–6] to improve the efficiency for computing 1D Sommerfeld integrals. The comparisons between the quasi-static and dynamic fields that are given in [4] show that the accuracy of the static approximation depends on the parameters in the problem. Therefore, the upper limit at which the integral in the AET is truncated also depends on the parameters in the problem. The most important parameter is the distance to the nearest interface. The authors of [5] demonstrate that the AET can be used to lower the value of the upper integration limit; however, no results are given for the amount of CPU time saved by using this method. The techniques illustrated in the above papers are less involved than the technique which will be developed in this paper, since they only deal with 1D integrals, but the idea is the same.

In [8], the author takes this procedure one step further. He applies an AET to the 1D Sommerfeld integral, and then also carries out the four-dimensional surface integration that is required to compute the asymptotic portion of the impedance element. This technique was developed for pulse basis functions.

An AET can also be applied to the 2D Sommerfeld integrals that are encountered in the spectral domain representation of the impedance elements [9, 10]. Pozar shows in [9] that a homogeneous-space term can be extracted from the integrand, thereby yielding a integral that converges more rapidly. We will refer to this method as the homogeneous-space term extraction technique (HSTET). Later, we will compare the AET that is developed in this paper with the HSTET.

We will make a number of references to [7, 11] in this paper; however, the material that is contained in these papers can also be found in [12].

2. BACKGROUND MATERIAL

As was demonstrated in [2], the angular integral in [2, (8)] can be decomposed in terms of a new integral $\mathscr{I}_3$ [2, (17)]. It was also noted in Appendix A.7 of [2] that the correct result for $\mathscr{I}_1$ is obtained if $\hat{\hat{\mathscr{I}}}_3$ is used in place of $\mathscr{I}_3$ in the expression for $\mathscr{I}_1$ [2, (20), (72)]. Expressing $\mathscr{I}_1$ in terms of $\hat{\hat{\mathscr{I}}}_3$ simplifies the expressions in the AET. Taking advantage of this, we can use the results in [2] to show that

$$\begin{aligned}\mathscr{I}_1(k_A, \lambda, x, 0, (0, 0, S_3, 1)) &= \tfrac{1}{8} \sum_{q=0}^{1} (-1)^q \{[2+4\cos^2(k_A d)]\, \hat{\hat{\mathscr{I}}}_3(k_A, \lambda, x, q2v, \mathbf{S}) \\ &\quad + \sum_{p=-2}^{2}{}' [-4\cos(k_A d)]^{2-|p|} \\ &\quad \times \hat{\hat{\mathscr{I}}}_3(k_A, \lambda, x+pd, q2v, \mathbf{S})\}|_{\mathbf{S}=(0,0,S_3,1)},\end{aligned} \tag{1}$$

where $\sum_p'$ means to sum over all p excluding $p=0$.

After decomposing the angular integral in terms of ILHIs, the series expansions that are derived in [7] are used in [2] to compute these ILHIs. These series expansions will also be used in the AET which is developed in the next section.

3. ASYMPTOTIC EXPANSION OF THE INTEGRAND

Before we start the analysis, it is useful to define the following quantities for $\mathbf{S}=(0, 0, S_3, 1)$:

$$\begin{aligned}\mathscr{I}_{14}^{(n)}(\lambda) = 2\lambda\{&[f_1^{(11)}(\lambda, 0) - f_2^{(11)}(\lambda, 0)]\, D_n|_{S_3=0} \\ &+ f_2^{(11)}(\lambda, 0)\, D_n|_{S_3=1}\}\end{aligned} \tag{2}$$

and

$$\mathscr{I}_{15}(\lambda, x, y) = \mathscr{I}_{14}^{(1)}(\lambda)\, \Re\{\mathscr{I}_7(\lambda, x, y, 0, 1)\} + \mathscr{I}_{14}^{(9)}(\lambda)\, \Re\left\{\mathscr{I}_4\left(\lambda, x, y, 0, \frac{k_A}{\lambda}\right)\right\} + \mathscr{I}_{14}^{(5)}(\lambda)\, \Re\left\{\mathscr{I}_{12}\left(\lambda, x, y, \frac{k_A}{\lambda}\right)\right\}. \quad (3)$$

The contribution to Z_{mn}, which is due to large values of the spectral variable λ, can be separated out from [2, (7)] by defining a new integral (see [2, Table III, (47), (48), (72), (73)], (1), (2), and (3))

$$Z_{mn}^a(L) = \frac{1}{8}\left[\frac{k_A}{v\pi \sin(k_A d)}\right]^2 \sum_{q=0}^{1} (-1)^q \int_L^\infty \Big\{\mathscr{I}_{15}(\lambda, x, q2v) \times [2 + 4\cos^2(k_A d)] + \sum_{p=-2}^{2}{}' [-4\cos(k_A d)]^{2-|p|} \times \mathscr{I}_{15}(\lambda, x + pd, q2v)\Big\}\, d\lambda, \quad (4)$$

where L is chosen large enough that the asymptotic expansions, which are derived in Appendix A, will provide the desired accuracy for the integrand.

We will start the analysis by looking at the asymptotic behavior of τ_{pq}. Referring to [2, (10)], we find that

$$\tau_{pq} = -j\lambda\sqrt{1 - k_{pq}^2/\lambda^2} \sim -j\lambda\left[1 - \frac{k_{pq}^2}{2\lambda^2} - \frac{k_{pq}^4}{8\lambda^4} + O(\lambda^{-6})\right]. \quad (5)$$

In order to find the behavior of the reflection coefficients for large values of λ, we can substitute (5) into [2, (12), (14)], yielding

$$|\Gamma_U^{(pq)}| \propto |\Gamma_V^{(pq)}| \propto |e^{-2p\lambda z_{pq}}|; \qquad \lambda \gg |k_{pq}|. \quad (6)$$

If we assume that only non-magnetic materials are present in the problem (i.e., $\mu_{pq} = \mu_0$), then we can use [2, (9), (16)] along with (6) to show that the asymptotic expansions,

$$f_1^{(11)}(\lambda, 0) \sim \frac{\omega\mu_0}{\lambda^2(\tau_{11} + \tau_{-11})}$$
$$f_2^{(11)}(\lambda, 0) \sim \frac{\omega\mu_0 \tau_{11}\tau_{-11}}{\lambda^2(\tau_{11}k_{-11}^2 + \tau_{-11}k_{11}^2)}, \quad (7)$$

hold when λ is large enough that

$$|e^{-2p\lambda z_{p1}}| < \tfrac{1}{2} \times 10^{-\mathrm{SD}}, \quad (8)$$

where SD is the desired number of significant digits. Furthermore, substituting (5) into (7) enables us to write

$$f_1^{(11)}(\lambda, 0) \sim \frac{j\omega\mu_0}{2\lambda^3}\left\{1 + \frac{(k_{11}^2 + k_{-11}^2)}{4\lambda^2} + \frac{(k_{11}^4 + k_{11}^2 k_{-11}^2 + k_{-11}^4)}{8\lambda^4}\right\}$$
$$f_2^{(11)}(\lambda, 0) \sim \frac{-j\omega\mu_0}{\lambda(k_{11}^2 + k_{-11}^2)}\left\{1 - \frac{(k_{11}^4 + k_{-11}^4)}{2\lambda^2(k_{11}^2 + k_{-11}^2)} - \frac{(k_{11}^4 + k_{11}^2 k_{-11}^2 + k_{-11}^4)}{8\lambda^4} + \frac{k_{11}^4 k_{-11}^4}{\lambda^4(k_{11}^2 + k_{-11}^2)^2}\right\}. \quad (9)$$

An asymptotic expansion for $\mathscr{I}_{14}^{(n)}(\lambda)$ can now be obtained, using [2, Table III], (2), and (9),

$$\mathscr{I}_{14}^{(1)}(\lambda) \sim \frac{-j\omega\mu_0}{\lambda^4(k_{11}^2 + k_{-11}^2)}\left\{c_0^{(1)} + \frac{c_1^{(1)}}{\lambda^2} + \frac{c_2^{(1)}}{\lambda^4}\right\}, \quad (10)$$

$$\mathscr{I}_{14}^{(5)}(\lambda) \sim \frac{j\omega\mu_0}{2\lambda^4 k_A^2(k_{11}^2 + k_{-11}^2)}\left\{c_0^{(5)} + \frac{c_1^{(5)}}{\lambda^2} + \frac{c_2^{(5)}}{\lambda^4}\right\}, \quad (11)$$

$$\mathscr{I}_{14}^{(9)}(\lambda) \sim \frac{j\omega\mu_0}{2\lambda^3 k_A^3(k_{11}^2 + k_{-11}^2)}\left\{c_0^{(9)} + \frac{c_1^{(9)}}{\lambda^2} + \frac{c_2^{(9)}}{\lambda^4}\right\}, \quad (12)$$

where

$$c_0^{(1)} = 1$$
$$c_1^{(1)} = 2k_A^2 - \frac{(k_{11}^4 + k_{-11}^4)}{2(k_{11}^2 + k_{-11}^2)}$$
$$c_2^{(1)} = k_A^2\left[3k_A^2 - \frac{(k_{11}^4 + k_{-11}^4)}{(k_{11}^2 + k_{-11}^2)}\right] - \frac{(k_{11}^4 + k_{11}^2 k_{-11}^2 + k_{-11}^4)}{8} + \frac{k_{11}^4 k_{-11}^4}{(k_{11}^2 + k_{-11}^2)^2}, \quad (13)$$

$$c_0^{(5)} = \frac{(k_{11}^2 + k_{-11}^2)}{2} - k_A^2$$
$$c_1^{(5)} = \frac{(k_{11}^2 + k_{-11}^2)^2}{8} + \frac{k_A^2(k_{11}^4 + k_{-11}^4)}{2(k_{11}^2 + k_{-11}^2)} - k_A^4$$
$$c_2^{(5)} = \frac{(k_{11}^4 + k_{11}^2 k_{-11}^2 + k_{-11}^4)[(k_{11}^2 + k_{-11}^2) + 2k_A^2]}{16} - \frac{k_A^2 k_{11}^4 k_{-11}^4}{(k_{11}^2 + k_{-11}^2)^2} + \frac{k_A^4(k_{11}^4 + k_{-11}^4)}{2(k_{11}^2 + k_{-11}^2)} - k_A^6, \quad (14)$$

$$c_0^{(9)}=\frac{(k_{11}^2+k_{-11}^2)}{2}+k_A^2$$

$$c_1^{(9)}=\frac{(k_{11}^2+k_{-11}^2)^2}{8}-\frac{k_A^2(k_{11}^4+k_{11}^2k_{-11}^2+k_{-11}^4)}{(k_{11}^2+k_{-11}^2)}+4k_A^4$$

$$c_2^{(9)}=\frac{(k_{11}^4+k_{11}^2k_{-11}^2+k_{-11}^4)[(k_{11}^2+k_{-11}^2)-2k_A^2]}{16} \quad (15)$$

$$+k_A^2\left[\frac{k_{11}^4k_{-11}^4}{(k_{11}^2+k_{-11}^2)^2}-\frac{(k_{11}^2+k_{-11}^2)^2}{8}\right]$$

$$-k_A^4\left[\frac{2(k_{11}^4+k_{-11}^4)}{(k_{11}^2+k_{-11}^2)}+\frac{(k_{11}^2+k_{-11}^2)}{2}\right]+8k_A^6.$$

The next step is to obtain an asymptotic expansion for $\mathscr{I}_{15}$, which when integrated from L to ∞, as is required in (4), can be represented in terms of special functions. This will involve finding expansions for $\mathscr{I}_7$, $\mathscr{I}_4$, and $\mathscr{I}_{12}$. Expansions for these three functions are obtained in Appendix A.

Referring to (3), (10)–(12), (56), (59), (60), (64), (66), (68), and (69), we find that in order to evaluate Z_{mn}^a (4) we need to evaluate integrals that have the general form

$$j_{m,n}(L,r)=\int_L^\infty \frac{J_n(\lambda r)}{\lambda^m}\,d\lambda. \quad (16)$$

If we carry out all of the expansions in λ to the same order as in (5), then we will need to compute $j_{m,n}(L,r)$ for $m=3, 4, ..., 10$ and $n=0, 1, ...$ Therefore, all that we need to do is find an efficient way to compute integrals which are of the form given above. Since we will need to compute a sequence of these integrals, it is beneficial to use a recurrence algorithm. We have found that the recurrence relation

$$j_{m,n}(L,r)-\frac{(n+1+m)}{(n+1-m)}j_{m,n+2}(L,r)=-\frac{2(n+1)}{(n+1-m)}\frac{J_{n+1}(Lr)}{rL^m} \quad (17)$$

is well suited for this purpose. The above recurrence relation can be obtained by rearranging [13, (11.3.6)]. Examining (17), we find that this recurrence relation decouples when $n+1-m=0$. At this point, the recurrence relation can be rewritten as

$$j_{m,m+1}(L,r)=\frac{J_m(Lr)}{rL^m}. \quad (18)$$

Before we use (17), we must determine in which direction the recurrence is stable. In Appendix B, we show that (17) can be used stably in the forward direction for all values of m and Lr. When the recurrence relation decouples at the point $n+1-m=0$, (18) can be used to restart the recurrence. Therefore, if we can find a way to obtain the starting functions $j_{m,0}(L,r)$ and $j_{m,1}(L,r)$ for $m=3, 4, ..., 10$, then we can use (17) in the forward direction to obtain $j_{m,n}(L,r)$ for $n=2, 3, ...$

Using [13, (11.3.4)], we can show that

$$j_{m,n}(L,r)-\frac{(m+n+1)}{r}j_{m+1,n+1}(L,r)=-\frac{J_{n+1}(Lr)}{rL^m}. \quad (19)$$

We are interested in using (19) for the two special cases where $n=-1$ and $n=0$:

$$j_{m,1}(L,r)+\frac{m}{r}j_{m+1,0}(L,r)=\frac{J_0(Lr)}{rL^m}$$
$$j_{m,0}(L,r)-\frac{m+1}{r}j_{m+1,1}(L,r)=-\frac{J_1(Lr)}{rL^m}. \quad (20)$$

This shows that once the two starting functions, $j_{m,0}(L,r)$ and $j_{m,1}(L,r)$, are obtained for one value of m, then the recurrence relations in (20) can be used to compute the starting functions for the other values of m. As is shown in Appendix B, the recurrence relations in (20) can be used stably in the forward direction to compute $j_{m,0}(L,r)$ for $m=m_{\max}+1$, $m_{\max}+2, ..., 10$, and backward recurrence can be used stably for $m=m_{\max}-1$, $m_{\max}-2, ..., 3$. Thus, if we can find some way to compute $j_{m_{\max},0}(L,r)$ and $j_{m_{\max},1}(L,r)$, then we can use (20) to compute the starting functions for the other values of m. This task is carried out in Appendix C.

4. COMPUTATION OF $\int_L^\infty \mathscr{I}_{15}(\lambda, x, y)\,d\lambda$

Now that we have developed all of the required tools, we can construct an efficient algorithm for the computation of $\int_L^\infty \mathscr{I}_{15}(\lambda, x, y)\,d\lambda$, where $\mathscr{I}_{15}$ is defined in (3). Once we can compute this integral, then (4) can be used to compute $Z_{mn}^a(L)$.

Up to this point, we have not specified a value for L. If we carry out all of the expansions to the same order of accuracy as in (5), then we will obtain SD significant digits in the expansions provided that L satisfies the following inequality:

$$\left[\frac{\max(k_{11}, k_{-11}, k_A)}{L}\right]^6<\frac{1}{2}\times 10^{-\mathrm{SD}}. \quad (21)$$

The inequality in (8) gives a second constraint for the choice for L:

$$e^{-2L\min(z_{11},\,-z_{-11})}<\tfrac{1}{2}\times 10^{-\mathrm{SD}}. \quad (22)$$

We will choose the minimum value of L that satisfies the above two inequalities.

The first integral that we will look at is (see (3))

$$\int_L^\infty \mathscr{I}_{14}^{(1)}(\lambda)\,\Re\{\mathscr{I}_7(\lambda, x, y, 0, 1)\}\,d\lambda. \tag{23}$$

Referring to (10), (16), and (56), we find that

$$\begin{aligned}\int_L^\infty \mathscr{I}_{14}^{(1)}(\lambda)\,\Re\{\mathscr{I}_7(\lambda, x, y, 0, 1)\}\,d\lambda \\ = \frac{-j2\pi\omega\mu_0}{(k_{11}^2+k_{-11}^2)}\sum_{i=0}^{2} c_i^{(1)}\Big\{ r j_{3+2i,1}(L, r) \\ -2y\sum_{k=0}^{\infty}(-1)^k \sin[(2k+1)\,\theta_0]\, j_{3+2i,2k+1}(L, r)\Big\}.\end{aligned} \tag{24}$$

This series will not converge until $2k+1 \gg Lr$ (see (76) and (78)). In order to investigate the convergence of (24), we can use (78) to show that

$$j_{3+2i,2k+1}(L, r) \sim \frac{1}{r}\left(\frac{r}{2}\right)^{3-2i} \frac{\Gamma(k-i-1/2)}{\Gamma(k+i+5/2)}; \quad 2k+1 \gg Lr. \tag{25}$$

This shows that when summing over k in (24), the series will converge more rapidly when $i=2$ than when $i=0$. Actually, we have found that there are numerical problems associated with summing the series over k for $i=0$ and $i=1$. Therefore, we will attack the problem from a different angle.

If we define

$$\mathscr{I}_{17}(x, y; n) = \int_L^\infty \frac{e^{j\lambda x} Je_0(j\cos\theta_0, \lambda r)}{\lambda^{n+1}}\,d\lambda, \tag{26}$$

then it can be shown that (see [2, Table III] and (10))

$$\begin{aligned}\int_L^\infty \mathscr{I}_{14}^{(1)}(\lambda)\,\Re\{\mathscr{I}_7(\lambda, x, y, 0, 1)\}\,d\lambda \\ = \frac{-j2\pi\omega\mu_0}{(k_{11}^2+k_{-11}^2)}\sum_{i=0}^{2} c_i^{(1)}\{ r j_{3+2i,1}(L, r) \\ - y\sin\theta_0\,\Re\{\mathscr{I}_{17}(x, y; 2+2i)\}\}.\end{aligned} \tag{27}$$

Now, comparing (24) with (27), we find that

$$\begin{aligned}\Re\{\mathscr{I}_{17}(x, y; 2+2i)\} \\ = \frac{2}{\sin\theta_0}\sum_{k=0}^{\infty}(-1)^k \\ \times \sin[(2k+1)\,\theta_0]\, j_{3+2i,2k+1}(L, r).\end{aligned} \tag{28}$$

Using numerical tests, we found that this expansion can be used to compute $\mathscr{I}_{17}(x, y; 6)$.

Next, we need to find an efficient way to compute $\mathscr{I}_{17}(x, y; 2+2i)$ for $i=0$ and $i=1$. We will develop a recurrence relation for this purpose. When $x \neq 0$, we can use integration by parts, where $u = Je_0(j\cos\theta_0, \lambda r)/\lambda^{n+1}$ and $dv = e^{j\lambda x}\,d\lambda$, and [2, (21)] to show that

$$\begin{aligned}\mathscr{I}_{17}(x, y; n) = \frac{j}{x}\Big\{\frac{e^{jLx}}{L^{n+1}} Je_0(j\cos\theta_0, Lr) - (n+1) \\ \times \mathscr{I}_{17}(x, y; n+1) + r j_{n+1,0}(L, r)\Big\}; \qquad x \neq 0.\end{aligned} \tag{29}$$

We only need to compute $\Re\{\mathscr{I}_{17}(x, y; n)\}$ for even values of n, therefore, we can use (29) to show that

$$\begin{aligned}\Re\{x^2\mathscr{I}_{17}(x, y; n) + (n+1)(n+2)\,\mathscr{I}_{17}(x, y; n+2)\} \\ = (n+1)\, r j_{n+2,0}(L, r) \\ + [(n+1)\,\Re\{e^{jLx} Je_0(j\cos\theta_0, Lr)\} \\ - Lx\,\Im\{e^{jLx} Je_0(j\cos\theta_0, Lr)\}]/L^{n+2}.\end{aligned} \tag{30}$$

When $x=0$, we can rewrite (29) as

$$\mathscr{I}_{17}(0, y; n) = \frac{1}{n}\left[\frac{1}{L^n} Je_0(0, Ly) + y j_{n,0}(L, r)\right]. \tag{31}$$

Before we can use (29), we need to perform a stability analysis. The homogeneous solution of (29) is given by [14]

$$\mathscr{I}_{17}^{(h)}(x, y; n) = \frac{(jx)^n}{\Gamma(n+1)}, \tag{32}$$

and an index of stability for forward recurrence is given by

$$\alpha_{17}(k, n) = \left|\frac{\mathscr{I}_{17}(x, y; k)\,\mathscr{I}_{17}^{(h)}(x, y; n)}{\mathscr{I}_{17}(x, y; n)\,\mathscr{I}_{17}^{(h)}(x, y; k)}\right| = \frac{\rho_{17}(n)}{\rho_{17}(k)}, \tag{33}$$

where

$$\rho_{17}(n) = \left|\frac{\mathscr{I}_{17}(x, y; 0)\,\mathscr{I}_{17}^{(h)}(x, y; n)}{\mathscr{I}_{17}(x, y; n)}\right|. \tag{34}$$

We need to obtain an approximation for $|\mathscr{I}_{17}(x, y; n)|$ before we can use (33). This can be accomplished by substituting [7, (77)] into (26):

$$|\mathscr{I}_{17}(x, y; n)| \sim \left| \csc\theta_0 \int_L^\infty \frac{e^{j\lambda x}}{\lambda^{n+1}}\, d\lambda - \sqrt{\frac{2}{\pi r}} \csc^2\theta_0 \int_L^\infty \lambda^{-n-3/2} \left[\cos\left(\lambda r - \frac{\pi}{4}\right) \times j\cos\theta_0 - \sin\left(\lambda r - \frac{\pi}{4}\right)\right] d\lambda \right|;$$

$$\min(Lr, Lr\,|\cos\theta_0 \pm 1|) \gg 0. \tag{35}$$

When $\min(Lr, Lr\,|\cos\theta_0 \pm 1|) \gg 1$, we can use (96) to show that

$$|\mathscr{I}_{17}(x, y; n)| \sim \left| \frac{\csc\theta_0}{xL^{n+1}} \right|. \tag{36}$$

Now, we can use (32), (34), and (36) to show that

$$\rho_{17}(n) \sim \frac{|Lx|^n}{\Gamma(n+1)}; \qquad |Lx| \gg 0. \tag{37}$$

We can also show that (37) still holds when $|\cos\theta_0| \approx 1$ by substituting [7, (80)), instead of [7, (77)], into (26).

Referring to (30), (37), and [7, (106)], we find that once $\mathscr{I}_{17}(x, y; 6)$ has been computed, we can stably use (29) in the backward direction to calculate $\mathscr{I}_{17}(x, y; 4)$ and $\mathscr{I}_{17}(x, y; 2)$ when $|Lx| > 7.0$ (see [14] for details of the stability analysis). When $|Lx| < 7.0$, we can still use backward recurrence, but accuracy will be lost. However, this loss of accuracy will not cause problems for the applications which are presented. Therefore, we can use the results in this section to compute the integral in (27).

The next step is to determine how to compute

$$\int_L^\infty \mathscr{I}_{14}^{(9)}(\lambda)\, \Re\left\{\mathscr{I}_4\left(\lambda, x, y, 0, \frac{k_A}{\lambda}\right)\right\} d\lambda$$
$$\int_L^\infty \mathscr{I}_{14}^{(5)}(\lambda)\, \Re\left\{\mathscr{I}_{12}\left(\lambda, x, y, \frac{k_A}{\lambda}\right)\right\} d\lambda. \tag{38}$$

When the inequality in (63) holds, we can use the asymptotic expansions in (59) and (60). On the other hand, when the inequality in (63) does not hold, we will have to use another method to compute (38). When $x = 0$, we can use the convergent series expansions in (64) and (66). Finally, when the inequality in (63) does not hold and $x \neq 0$, we will make use of (68) and (69). In order to do this, we split the integrals in (38) into two pieces,

$$\int_L^\infty \{\ \}\, d\lambda = \int_L^{L'} \{\ \}\, d\lambda + \int_{L'}^\infty \{\ \}\, d\lambda, \tag{39}$$

where

$$L' = \frac{2r}{x^2} \ln\left(2\sqrt{2} \times 10^{\mathrm{SD}}\right). \tag{40}$$

For values of x that are of the same order as y, L' will be the smallest value of λ that will satisfy the inequality in (63) for $\lambda \gg k_A$. Therefore, the second integral on the right-hand side of (39) can be computed using (59) and (60). Now, if we substitute the convergent series expansions, (68) and (69), into the first integral on the right-hand side of (39), then the evaluation of this integral involves the computation of

$$\hat{\jmath}_{m,n}(L, L', r) = j_{m,n}(L', r) - j_{m,n}(L, r). \tag{41}$$

We can use a modified version of (17) to compute $\hat{\jmath}_{m,n}(L, L', r)$,

$$\hat{\jmath}_{m,n}(L, L', r) - \frac{(n+1+m)}{(n+1-m)} \hat{\jmath}_{m,n+2}(L, L', r) = \frac{2(n+1)}{r(n+1-m)} \left[\frac{J_{n+1}(Lr)}{L^m} - \frac{J_{n+1}(L'r)}{(L')^m}\right]; \tag{42}$$

however, we must first perform a stability analysis. Since we have not changed the homogeneous equation, (72) and (73) will still hold. When $n \ll Lr$, we can obtain an approximation for $\hat{\jmath}_{m,n}(L, L', r)$ by substituting (76) into (41). Since $\hat{\jmath}_{m,n}(L, L', r)$ has the same kind of behavior as $j_{m,n}(L, r)$ when $n \ll Lr$, we conclude that (42) can be used stably in the forward direction. On the other hand, when $n \gg L'r$, we can use [13, (9.3.1)] to show that

$$\hat{\jmath}_{m,n}(L, L', r) \sim \left(\frac{er}{2n}\right)^n \frac{[(L')^{n+1-m} - L^{n+1-m}]}{\sqrt{2\pi n}\,(n+1-m)}. \tag{43}$$

Therefore, we find that (42) must be used in the backward direction when $n \gg L'r$. We have found using numerical tests that forward recurrence can be used to compute $\hat{\jmath}_{m,n}(L, L', r)$ for $n = 0, 1, ..., m$, and backward recurrence can be used for $n = m + 1, ...$

5. REFLECTION PROPERTIES OF $\int_L^\infty \mathscr{I}_{15}(\lambda, x, y)\, d\lambda$

We will look at the reflection properties for $\int_L^\infty \mathscr{I}_{15}(\lambda, x, y)\, d\lambda$ in this section. However, first we will look at the reflection properties of $\mathscr{I}_{15}$. Using the results in [2], (2), and (3), we find that

$$\mathscr{I}_{15}(\lambda, -x, y) = \mathscr{I}_{15}(\lambda, x, y) - \frac{2\pi\lambda}{\sqrt{\lambda^2 - k_A^2}} \mathscr{I}_{14}^{(9)}(\lambda) \times \{\sin(\lambda F_+ r) + \sin(\lambda F_- r)\} - \frac{2\pi\lambda^3 r}{(\lambda^2 - k_A^2)} \mathscr{I}_{14}^{(5)}(\lambda) \times \{\sqrt{1 - F_+^2}\cos(\lambda F_+ r) + \sqrt{1 - F_-^2}\cos(\lambda F_- r)\}. \tag{44}$$

This expression can be further simplified by applying [13, (4.3.34)-(4.3.37)]

$$\begin{aligned}
&\mathscr{I}_{15}(\lambda, -x, y)\\
&\quad = \mathscr{I}_{15}(\lambda, x, y) + \frac{4\pi\lambda}{\sqrt{\lambda^2 - k_A^2}} \mathscr{I}_{14}^{(9)}(\lambda) \sin(k_A x)\\
&\quad \times \cos(y\sqrt{\lambda^2 - k_A^2}) - \frac{4\pi\lambda^2}{(\lambda^2 - k_A^2)\, \mathscr{I}_{14}^{(5)}(\lambda)}\\
&\quad \times \{k_A y \sin(k_A x) \sin(y\sqrt{\lambda^2 - k_A^2})\\
&\quad + x\sqrt{\lambda^2 - k_A^2} \cos(k_A x) \cos(y\sqrt{\lambda^2 - k_A^2})\}. \qquad (45)
\end{aligned}$$

When $y \neq 0$, we can integrate both sides of (45) with respect to λ and apply the change of variables, $u = y\sqrt{\lambda^2 - k_A^2}$, thereby obtaining (see (10)–(12))

$$\begin{aligned}
&\int_L^\infty \mathscr{I}_{15}(\lambda, -x, y)\, d\lambda\\
&\quad = \int_L^\infty \mathscr{I}_{15}(\lambda, x, y)\, d\lambda + \frac{j2\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{-11}^2)}\\
&\quad \times \sum_{i=0}^{2} \left\{ \left[\frac{\sin(k_A x)}{k_A} c_i^{(9)} - x\cos(k_A x)\, c_i^{(5)} \right] \right.\\
&\quad \times \frac{1}{y} \int_{u'}^\infty \left(\frac{y}{u}\right)^{2i+3} \frac{\cos u\, du}{[1 + (k_A y/u)^2]^{i+3/2}}\\
&\quad - k_A \sin(k_A x)\, c_i^{(5)} \int_{u'}^\infty \left(\frac{y}{u}\right)^{2i+4}\\
&\quad \left. \times \frac{\sin u\, du}{[1 + (k_A y/u)^2]^{i+3/2}} \right\}; \qquad y \neq 0, \qquad (46)
\end{aligned}$$

where $u' = y\sqrt{L^2 - k_A^2}$. When $\lambda \gg k_A$, we can use [13, (3.6.9), (6.1.22), and (6.5.3)] to obtain the expansion

$$\begin{aligned}
&\int_L^\infty \mathscr{I}_{15}(\lambda, -x, y)\, d\lambda\\
&\quad = \int_L^\infty \mathscr{I}_{15}(\lambda, x, y)\, d\lambda\\
&\quad + \frac{j2\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{-11}^2)} \sum_{i=0}^{2} \sum_{n=0}^{\infty} (-k_A^2)^n\\
&\quad \times y^{2n+2i+3} \frac{\Gamma(n+i+3/2)}{\Gamma(n+1)\,\Gamma(i+3/2)}\\
&\quad \times \left\{ \left[\frac{\sin(k_A x)}{k_A} c_i^{(9)} - x\cos(k_A x)\, c_i^{(5)} \right] \right.\\
&\quad \times \frac{1}{y} \int_{u'}^\infty \frac{\cos u\, du}{u^{2n+2i+3}} - k_A y \sin(k_A x)\, c_i^{(5)}\\
&\quad \left. \times \int_{u'}^\infty \frac{\sin u\, du}{u^{2n+2i+4}} \right\}; \qquad y \neq 0. \qquad (47)
\end{aligned}$$

Referring to (91), we find that the integrals in (47) can be represented as incomplete gamma functions. Actually, the results in Appendix C can be modified and used to compute the incomplete gamma functions in (47).

We will have to obtain another expansion for the special case $y = 0$. We start by integrating both sides of (45), where y is set equal to zero:

$$\begin{aligned}
&\int_L^\infty \mathscr{I}_{15}(\lambda, -x, 0)\, d\lambda\\
&\quad = \int_L^\infty \mathscr{I}_{15}(\lambda, x, 0)\, d\lambda + \frac{j2\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{-11}^2)} \sum_{i=0}^{2}\\
&\quad \times \left[\frac{\sin(k_A x)}{k_A} c_i^{(9)} - x\cos(k_A x)\, c_i^{(5)} \right]\\
&\quad \times \int_L^\infty \frac{d\lambda}{\lambda^{2i+3}\sqrt{1 - (k_A/\lambda)^2}}. \qquad (48)
\end{aligned}$$

When $\lambda \gg k_A$, we can use [13, (3.6.9) and (6.1.22)] to expand the square root in a power series, and then we can integrate term-by-term, yielding

$$\begin{aligned}
&\int_L^\infty \mathscr{I}_{15}(\lambda, -x, 0)\, d\lambda\\
&\quad = \int_L^\infty \mathscr{I}_{15}(\lambda, x, 0)\, d\lambda + \frac{j\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{-11}^2)}\\
&\quad \times \sum_{i=0}^{2} \left[\frac{\sin(k_A x)}{k_A} c_i^{(9)} - x\cos(k_A x)\, c_i^{(5)} \right]\\
&\quad \times \sum_{n=0}^{\infty} \frac{(k_A)^{2n}\, \Gamma(n+1/2)}{\Gamma(n+1)\,\Gamma(1/2)(n+i+1)\, L^{2n+2i+2}}. \qquad (49)
\end{aligned}$$

6. COMPARISON WITH THE HSTET

Now we will compare the AET which is developed in this paper with the HSTET which is used in [9, 10]. In order to apply the results in this paper, the lower limit of integration in (4), L, must satisfy the two inequalities in (21) and (22). We would like to choose L as small as possible, since it corresponds to the upper limit of integration in the numerical integration routine that is used to compute [2, (7)]. The inequality in (22) is required for the approximations in (7) to hold. A similar approximation is required in the HSTET; therefore, the inequality in (22) applies to both methods.

If we carry out all of the expansions in this paper to the same order of λ as in (5), then we will obtain SD significant digits in the approximation for the integrand of (4) provided that the inequality in (21) is satisfied. In [9], the asymptotic form for Q [9, (8)] can be obtained from [9, (6)] by using what is equivalent to the first term in the expansion (5)

(Pozar's k_1 and k_2 correspond to τ_{-11} and τ_{11}, respectively). Therefore (see [9, (8) and (13)]),

$$\frac{Q - Q^h}{Q} < \frac{1}{2} \times 10^{-\mathrm{SD}}, \tag{50}$$

for all $\beta > L$ provided that

$$\left[\frac{k_0\sqrt{\varepsilon_r}}{L}\right]^2 < \frac{1}{2} \times 10^{-\mathrm{SD}}. \tag{51}$$

The two inequalities in (21) and (51) can be used to find the approximate upper limit of integration, L, for the numerical integration routine. The inequality in (21) corresponds to a relative error bound for Z^a_{mn},

$$\frac{Z^a_{mn} - \widetilde{Z^a_{mn}}}{Z^a_{mn}} < \frac{1}{2} \times 10^{-\mathrm{SD}}, \tag{52}$$

where $\widetilde{Z^a_{mn}}$ is the approximation which is obtained for Z^a_{mn} by using the results in this paper. A more appropriate error bound would be

$$\frac{Z^a_{mn} - \widetilde{Z^a_{mn}}}{Z_{mn}} < \frac{1}{2} \times 10^{-\mathrm{SD}}; \tag{53}$$

however, this error bound is more difficult to use and does not provide insight into the problem. Therefore, we will use (21) and (51) to compare the two methods. Table I shows that carrying out more terms in the expansions, as is done in the AET, significantly reduces the ratio $[L/k_m]$ (where $k_m = \max(k_{11}, k_{-11}, k_A)$ and $k_m = k_0\sqrt{\varepsilon_r} = \max(k_{11}, k_{-11})$ for the AET and the HSTET, respectively) which is required to satisfy (21) and (51). Therefore, a smaller upper limit of integration and a correspondingly lower number of sample points will be required if the numerical integration routine is used in conjunction with the AET, instead of the HSTET, to compute the semi-infinite integral in [2, (7)]. In order to ensure that the desired accuracy is achieved, we will use a value of L that is one and one-half times greater than the minimum value of L which is found using (21) and (22).

The HSTET also requires that the expansion mode wavenumber, k_A (see [2, (5)]), be chosen as

$$k_A = k_e = \omega\sqrt{\mu_0(\varepsilon_{11} + \varepsilon_{-11})/2}. \tag{54}$$

As Pozar points out in [9], "The use of k_e as the wavenumber for the PWS modes, however, is sometimes a disadvantage in terms of overall convergence of the moment method solution for microstrip patches. As discussed in [15] and [16], the number of expansion modes needed for convergence of the input impedance of patches can be significantly reduced if an expansion mode wavenumber is chosen to correspond to the effective dielectric constant of the microstrip medium. This wavenumber is basically the same as k_e for thin substrates, but may differ for thick substrates, in which case the moment method solution using k_e will require a larger number of expansion modes for good results. In other words, the number of expansion modes needed for a given accuracy depends on the choice of the expansion mode wavenumber, and k_e is not always the optimum choice." In comparison, we are free to chose the optimal value for k_A in the AET.

When the HSTET is applied to a thin dipole antenna, Z^h_{mn} can be represented solely in terms of exponential integrals and other elementary functions [10, (18)]. However, an integration of the filamentary PWS modes across the width of the dipole is required for wide dipole antennas. On the other hand, the expansions which are given in this paper can be used for dipoles of any width; therefore, no numerical integration is required with the AET to obtain an approximation for Z^a_{mn}.

As we have shown in this section, there are a number of advantages associated with using the AET as compared with the HSTET. The major disadvantage of the AET lies in the complexity of the expansions.

TABLE I

$[L/k_m]$ Required for a Given Accuracy

	No. of significant digits SD			
	2	3	4	5
$\left[\frac{k_m}{L}\right]^6 < \frac{1}{2} \times 10^{-\mathrm{SD}}$	2.42	3.55	5.21	7.65
$\left[\frac{k_m}{L}\right]^2 < \frac{1}{2} \times 10^{-\mathrm{SD}}$	14.1	44.8	141.0	447.0

7. NUMERICAL RESULTS

In order to demonstrate the power of the AET, we will use it in the analysis of a printed strip dipole antenna in a layered medium. In [2], the technique of decomposing the angular integral, of the 2D Sommerfeld integrals, in terms of ILHIs was used to obtain the current distribution on a printed strip dipole antenna in a hyperthermia applicator. It was demonstrated in [2] that significant improvements in the computational efficiency can be achieved by employing this technique instead of using a numerical integration routine to compute the angular integral. Now we will show that application of the AET dramatically improves both the computational accuracy and efficiency of the outer semi-infinite integral in the 2D Sommerfeld integrals.

The hyperthermia applicator which will be used to demonstrate the AET is described in [2, Section 3]. We will use the same parameters as were used in [2] so that we can compare the results obtained in this paper with those obtained in [2]. The amount of computation time required to solve for the current distribution on the dipole antenna when the AET is employed is given in Table II.

When brute force numerical integration is used to compute the outer semi-infinite integrals, the integration procedure must be truncated at some upper limit L. In [2], this value of L was chosen by trial and error. This procedure is very inefficient. When the AET is employed, on the other hand, the truncation point for the numerical integration is chosen to be one and one-half times greater than the minimum value of L required in (21) and (51). For the problem under consideration, $L = 1485.52$. This truncation point is much smaller than those required in [2], therefore, fewer integration points will be required for the computation of the semi-infinite integral when the AET is employed. We will see that this leads to dramatic reductions in the computation time.

The first column in Table II shows the number of basis functions that were used on the antenna. Antennas of two different lengths were investigated. The results for an antenna at the first resonance are given in columns two through four. Columns five through seven contain the results for an antenna length corresponding to the third resonance. The columns headed by "$Z^a_{mn}(L)$" show the amount of computation time required to compute the asymptotic portion of the impedance elements. For the cases under consideration, the AET always took less than $\frac{1}{8}$ of the total computation time. A comparison between columns two and five shows that the AET is more efficient for the longer antenna. The reason for this is that the inequality in (63) is satisfied more often for the longer antenna since the values of r are larger for the longer antenna. When (63) is satisfied, the expansions in (64) and (66) can be used to compute the integrals in (38). On the other hand, when (63) is not satisfied, the integrals must be split into two pieces (see (39)) and a more time consuming computational procedure must be applied (see Section 4).

TABLE II

Typical CPU Times for the Computation of the Elements in the Impedance Matrix for PWS Basis Functions

No. of basis functions	Antenna length					
	$2l = 2.34$ cm			$2l = 7.76$ cm		
	$Z^a_{mn}(L)$	ILHIs	D01AJF	$Z^a_{mn}(L)$	ILHIs	D01AJF
1	0.54 s	8.38 s	8.80 s	0.26 s	14.82 s	49.00 s
3	1.68	16.02	20.38	0.86	30.74	97.42
5	2.62	26.64	34.70	1.10	45.10	163.66
7	3.78	36.92	48.04	1.78	60.44	236.40
9	4.96	43.98	58.94	2.06	72.16	283.14

The outer semi-infinite integrals, which were actually truncated at L, were computed using a version of D01AJF which was modified to compute complex valued integrals (see [17]). The inner angular integrals were computed using one of two methods. The technique of decomposing the angular integral in terms ILHIs [2] was used to obtain the results in the columns labeled "ILHIs"; and the results listed in the columns headed by "D01AJF" were obtained by applying the numerical integration routine D01AJF from the NAG library to the computation of the inner angular integral. Reference to [2] shows that the NAG adaptive quadrature routine D01AKF was used instead of D01AJF in that paper. The reason for this is we found that D01AJF worked better for small values of L and D01AKF was better suited for large values of L.

A comparison between columns three and four, or six and seven shows that the decomposition in terms of ILHIs provides a more efficient way to compute the angular integrals. This is especially true in the case of the longer antenna. The advantage of employing the AET is demonstrated by comparing the results in Table II and [2, Table I]. The AET not only provides a dramatic improvement in the computational efficiency, but it also allows for the accurate computation of the highly oscillatory integrals which the brute force double numerical integration routine could not handle (see [2, Table I]).

8. CONCLUSION

In conclusion, we have demonstrated that the asymptotic portion of the impedance elements (4) can be represented as a series of special functions which have the general form of (16). In addition, we have shown that recurrence can be used to compute the special functions (16) that are required. The application of this method to the analysis of a printed strip dipole antenna in a layered medium demonstrated that this novel AET provides an accurate and efficient way to compute the elements in the impedance matrix. Also, the computational efficiency can be further improved by applying the techniques which were developed in [2] to the inner angular integral.

As was mentioned in [2], the techniques which have been developed in this paper are not restricted to PWS basis functions. They can be used with any basis functions which result in angular integrals which have the general form (1).

APPENDIX A: EXPANSIONS FOR $\mathscr{I}_7$, $\mathscr{I}_4$, AND $\mathscr{I}_{12}$

We will first concentrate on finding an expansion for $\mathscr{I}_7$. If we substitute [7, (58)] into the expression for $\mathscr{I}_7$ [2, Table III], then we obtain the convergent series expansion,

$$\Re\{\mathscr{I}_7(\lambda, x, y, 0, 1)\} = 2\pi\lambda \left\{ rJ_1(\lambda r) - y\Re\left\{ e^{j\lambda x} - \sum_{k=0}^{\infty} (je^{j\theta_0})^k \varepsilon_k J_k(\lambda r) \right\}\right\}. \tag{55}$$

We can use [18, (8.511.4)] to rewrite the above equation as

$$\Re\{\mathscr{I}_7(\lambda, x, y, 0, 1)\} = 2\pi\lambda \left\{ rJ_1(\lambda r) - 2y \sum_{k=0}^{\infty} (-1)^k \times \sin[(2k+1)\theta_0] J_{2k+1}(\lambda r) \right\}. \tag{56}$$

Next we will obtain expansions for $\mathscr{I}_4$ and $\mathscr{I}_{12}$. Since we are looking for an expansion which holds for large values of λ, we will make use of [7, (57)]. However, before we proceed with finding the asymptotic expansions, we will derive some useful intermediate results.

Since we will be using [7, (57)], we will encounter terms which have the form (see [2, Table III, (47), (73)])

$$(\sqrt{1-P^2})^{-c} (\sqrt{1-F_{\pm}^2})^{-b} = \frac{(\sqrt{1-P^2})^{-(b+c)}}{\cos^b \theta_0} \left[1 \pm \frac{P \tan \theta_0}{\sqrt{1-P^2}} \right]^{-b}. \tag{57}$$

When $P = k_A/\lambda \ll 1$ and $P \tan \theta_0 \ll 1$, we can use ([13], (3.6.9) and (6.1.22)) to obtain the series expansion

$$(\sqrt{1-P^2})^{-c} (\sqrt{1-F_{\pm}^2})^{-b} = \frac{\cos^{-b}\theta_0}{\Gamma(b)} \sum_{m=0}^{\infty} \frac{(\mp P \tan \theta_0)^m}{\Gamma((m+b+c)/2)} \frac{\Gamma(m+b)}{\Gamma(m+1)} \times \sum_{n=0}^{\infty} P^{2n} \frac{\Gamma(n+(m+b+c)/2)}{\Gamma(n+1)}; \qquad c \geqslant 0, b \geqslant 0. \tag{58}$$

$$\Re\left\{\mathscr{I}_4\left(\lambda, x, y, 0, \frac{k_A}{\lambda}\right)\right\} \sim -2\pi \sum_{k=0}^{\infty} \left[\frac{2r}{k_A x^2}\right]^k \sum_{m=0}^{\infty} \left(\frac{2y}{x}\right)^{2m} \times \sum_{n=0}^{\infty} \left[\left(\frac{y}{x}\right)^2 \left(\frac{2k}{2m+1}+1\right)+1\right] \times \frac{\Gamma(m+k+1/2)\,\Gamma(n+m+k+1)}{\Gamma(1/2)\,\Gamma(2m+1)\,\Gamma(k+1)\,\Gamma(n+1)} J_k(\lambda r) \times \left(\frac{k_A}{\lambda}\right)^{k+2m+2n+1} \tag{59}$$

$$\Re\left\{\frac{\mathscr{I}_{12}(\lambda, x, y, k_A/\lambda)}{\lambda}\right\} \sim -2\pi r \sum_{k=1}^{\infty} \left[\frac{2r}{k_A x^2}\right]^k \sum_{m=0}^{\infty} \left(\frac{2y}{x}\right)^{2m} \times \sum_{n=0}^{\infty} \left[\frac{k}{m+k}\right] \frac{\Gamma(m+k+1/2)\,\Gamma(n+m+k+1)}{\Gamma(1/2)\,\Gamma(2m+1)\,\Gamma(k+1)\,\Gamma(n+1)} \times J_{k+1}(\lambda r) \left(\frac{k_A}{\lambda}\right)^{k+2m+2n} \tag{60}$$

Referring to (59) and (60), we find that these asymptotic expansions will only be useful when $\lambda \gg k_A$, $\lambda x \gg 2yk_A$, and $\lambda x^2 \gg 2r$. Actually, we can use [7, (76)] to show that the asymptotic factorial-Neumann series expansion, which was used to obtain (59) and (60), will provide the desired accuracy if

$$\frac{1}{2} \times 10^{-\mathrm{SD}} \left| e^{az} \left[Je_0(a, z) - \frac{1}{\sqrt{a^2+1}} \right] \right| > \frac{2}{|a^2+1| \sqrt{\pi z}} e^{-z|a^2+1|^{1/2}} \max(1, |a|). \tag{61}$$

If we modify [7, (77)], then we find that

$$\left| e^{az} \left[Je_0(a, z) - \frac{1}{\sqrt{a^2+1}} \right] \right| \sim \sqrt{\frac{2}{\pi z}} \frac{\max(1, |a|)}{|a^2+1|}; \qquad \min(z, z\,|a \pm j|) \gg 0. \tag{62}$$

Combining (61) and (62), we find that (59) and (60) will provide the desired accuracy provided the following inequality holds:

$$\lambda r\, |1 - F_{\pm}^2| > 2 \ln[2\sqrt{2} \times 10^{\mathrm{SD}}]. \tag{63}$$

The above inequality shows that we need to find other expansions for $\mathscr{I}_4$ and $\mathscr{I}_{12}$ which will work when x is small.

We will first look at the special case $x = 0$. For this case, we can use [7, (29); 2, Table III, (47), (73)] to show that

$$= -2\pi \Gamma\left(\frac{3}{2}\right) k_A y \sum_{k=0}^{\infty} \left[\frac{yk_A^2}{2\lambda}\right]^k \frac{J_{k+1}(\lambda y)}{\Gamma(k+3/2)} \tag{64}$$

$$\Re\left\{\frac{\mathscr{I}_{12}(\lambda, 0, y, k_A/\lambda)}{\lambda}\right\} = \frac{2\pi y}{1-(k_A/\lambda)^2} \left\{ J_1(\lambda y) - 2\Gamma\left(\frac{3}{2}\right) \sum_{k=0}^{\infty} \left[\frac{yk_A^2}{2\lambda}\right]^{k+1} \frac{J_k(\lambda y)}{\Gamma(k+3/2)} \right\}. \tag{65}$$

Also, when $\lambda \gg k_A$, we can use ([13], (3.6.9)) to rewrite (65) as

$$\Re\left\{\frac{\mathscr{I}_{12}(\lambda, 0, y, k_A/\lambda)}{\lambda}\right\}$$
$$\sim 2\pi y\left[1+\frac{k_A^2}{\lambda^2}+\frac{k_A^4}{\lambda^4}\right]\left\{J_1(\lambda y) - 2\Gamma\left(\frac{3}{2}\right)\sum_{k=0}^{\infty}\left[\frac{yk_A^2}{2\lambda}\right]^{k+1}\frac{J_k(\lambda y)}{\Gamma(k+3/2)}\right\}. \quad (66)$$

Now we will develop an expansion which can be used when the inequality in (63) does not hold and $x \neq 0$. This time we will use [7, (58)]. Once again it is beneficial to derive some intermediate results. First, the terms that we will encounter in the Neumann series expansion can be rewritten in the following form by using [13, (3.6.8), (6.1.21)]:

$$(\sqrt{1-P^2})^c\left(1-\frac{jP}{\sqrt{1-P^2}}\right)^b = \Gamma(b+1)\sum_{m=0}^{\infty}\frac{(-j)^m\,\Gamma((c-m)/2+1)}{\Gamma(m+1)\,\Gamma(b-m+1)} \times\sum_{n=0}^{\infty}\frac{(-1)^n\,(k_A/\lambda)^{2n+m}}{\Gamma(n+1)\,\Gamma((c-m)/2-n+1)};\ c\geqslant b\geqslant 0. \quad (67)$$

This result can now be used along with [7, (58); 2, Table III, (47), (73); 13, (3.6.9)] to obtain the expansions,

$$\Re\left\{\mathscr{I}_4\left(\lambda, x, y, 0, \frac{k_A}{\lambda}\right)\right\} = \pi\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\cos k\theta_0(k-2m)(k+1-2(m+n)) \times\frac{(-1)^{k+m+n}\,2\Gamma(k+1)\,\Gamma((k+1)/2-m)}{\Gamma(2m+2)\,\Gamma(k-2m+1)\,\Gamma(n+1)\,\Gamma((k+3)/2-m-n)} \times\left[1+\frac{k_A^2}{2\lambda^2}+\frac{3k_A^4}{8\lambda^4}\right]\left(\frac{k_A}{\lambda}\right)^{2m+2n+1}J_k(\lambda r) \quad (68)$$

$$\Re\left\{\frac{\mathscr{I}_{12}(\lambda, x, y, k_A/\lambda)}{\lambda}\right\} = 2\pi\left\{rJ_1(\lambda r)+\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\left(\frac{k_A}{\lambda}\right)^2 y\sin k\theta_0 \times(k-2m)\left(\frac{k+1}{2}-m-n\right) + x\cos k\theta_0\left(\frac{k+1}{2}-m\right)(2m+1)\right] \times\frac{(-1)^{k+m+n}\,\varepsilon_k\Gamma(k+1)\,\Gamma((k+1)/2-m)}{\Gamma(2m+2)\,\Gamma(k-2m+1)\,\Gamma(n+1)\,\Gamma((k+3)/2-m-n)} \times\left[1+\frac{k_A^2}{\lambda^2}+\frac{k_A^4}{\lambda^4}\right]\left(\frac{k_A}{\lambda}\right)^{2m+2n}J_k(\lambda r)\right\}. \quad (69)$$

APPENDIX B: STABILITY ANALYSIS FOR $j_{m,n}(L, r)$

We will first use the techniques in [14] to perform a stability analysis on (17). When n is an even integer, (17) takes on the general form

$$j_{m,2n}(L, r)+d^{(1)}_{m,2n}\,j_{m,2(n+1)}(L, r) = f^{(1)}_{m,2n};\qquad n=0, 1, ..., \quad (70)$$

where

$$d^{(1)}_{m,2n} = -\frac{(2n+1+m)}{(2n+1-m)}$$
$$f^{(1)}_{m,2n} = -\frac{2(2n+1)}{(2n+1-m)}\,\frac{J_{2n+1}(Lr)}{rL^m}. \quad (71)$$

The recurrence relation (70) behaves differently for even and odd values of m. When m is odd, the recurrence relation decouples at the point $2n+1-m=0$. On the other hand, when m is even the recurrence relation does not decouple. Since (70) behaves differently for even and odd values of m, we will have to handle the stability analysis for these two cases separately.

We will first handle the case for even values of m. For this case, we can use [13, (6.1.22); 14, (A.24)] to show that the homogeneous solution for (70) is given by

$$j^{(1h)}_{m,2n} = \prod_{k=0}^{n-1}[-d^{(1)}_{m,2k}]^{-1} = (-1)^n\frac{[\Gamma((m+1)/2)]^2}{\Gamma((m+1)/2-n)\,\Gamma((m+1)/2+n)};$$
$$m \text{ even, or } m \text{ odd and } 0\leqslant 2n\leqslant m-3. \quad (72)$$

It is slightly more difficult, however, to find the homogeneous solution for (70) for odd values of m. When $0\leqslant 2n\leqslant m-3$, we can once again directly apply [13, (6.1.22); 14, (A.24)], since the forward recurrence starts at $m=0$. Doing this, we find that (72) still holds for this case. On the other hand, when m is odd and $2n\geqslant m+1$, we will have to modify [14, (A.24)], since the forward recurrence will start at $2n=m+1$ (see (18)). This time we find that (see [14])

$$j^{(1h)}_{m,2n} = \prod_{k=(m+1)/2}^{n-1}[-d^{(1)}_{m,2k}]^{-1} = \frac{\Gamma(n+(1-m)/2)\,\Gamma(m+1)}{\Gamma(n+(m+1)/2)};$$
$$m \text{ odd},\ 2n\geqslant m+1. \quad (73)$$

In order to determine the stability of (70), we make use of the index of stability (see [14, (2.17)],

$$\alpha^{(1)}(2k, 2n) = \left| \frac{j_{m,2k}(L, r)\, j^{(1h)}_{m,2n}}{j_{m,2n}(L, r)\, j^{(1h)}_{m,2k}} \right| = \frac{\rho^{(1)}_{m,2n}}{\rho^{(1)}_{m,2k}}, \tag{74}$$

where

$$\rho^{(1)}_{m,2n} = \left| \frac{j_{m,0}(L, r)\, j^{(1h)}_{m,2n}}{j_{m,2n}(L, r)} \right|. \tag{75}$$

Before we can use (74), we need to obtain an approximation for $j_{m,n}(L, r)$. When $n \ll Lr$, we can use the first term in [13, (9.2.1)] to show that

$$j_{m,n}(L, r) \sim -\sqrt{\frac{2}{\pi r}}\, \frac{\sin(Lr - n\pi/2 - \pi/4)}{rL^{m+1/2}}; \qquad n \ll Lr. \tag{76}$$

On the other hand, when $n \gg Lr$ and $n + 1 > m > \frac{1}{2}$, we can split the integral into two pieces:

$$j_{m,n}(L, r) = \int_0^{\infty} \frac{J_n(r\lambda)}{\lambda^m}\, d\lambda - \int_0^{L} \frac{J_n(r\lambda)}{\lambda^m}\, d\lambda. \tag{77}$$

Now we can use [13, (9.3.1); 18, (6.561.14)] to show that

$$j_{m,n}(L, r) \sim r^{m-1} \left[\left(\frac{1}{2}\right)^m \frac{\Gamma((n+1-m)/2)}{\Gamma((n+1+m)/2)} - \left(\frac{eLr}{2n}\right)^n \frac{(Lr)^{1-m}}{\sqrt{2\pi n}\,(n+1-m)} \right];$$
$$n \gg Lr,\ n + 1 > m > \tfrac{1}{2}. \tag{78}$$

We are finally at a point where we can calculate the index of stability for (70). We will first handle the case where m is an even integer, or m is an odd integer and $0 \leqslant 2n \leqslant m - 3$. When $2n \gg Lr$ we can use (72), (75), (78), and [13, (6.1.17)] to show that

$$\rho^{(1)}_{m,2n} \sim \left(\frac{2}{Lr}\right)^{m+1/2} \frac{|\sin(Lr - \pi/4)|}{\pi^{3/2}} \times \left[\Gamma\left(\frac{m+1}{2}\right) \right]^2; \qquad m \text{ even},\ 2n \gg Lr. \tag{79}$$

On the other hand, when $2n \ll Lr$, we can use (72), (75), and (76) to show that

$$\rho^{(1)}_{m,2n} \sim \frac{[\Gamma((m+1)/2)]^2}{\Gamma((m+1)/2 - n)\, \Gamma((m+1)/2 + n)};$$
$$m \text{ even or } m \text{ odd and } 0 \leqslant 2n \leqslant m - 3,\ 2n \ll Lr. \tag{80}$$

Therefore, if we assume that an initial value, $j_{m,2k}(L, r)$, is known, then an index of stability for the forward computation of $j_{m,2n}(L, r)$ from $j_{m,2k}(L, r)$ can be obtained by substituting (79) or (80) into (74). For the purposes of this paper, we will be interested in computing $j_{m,n}(L, r)$ for $m = 3, 4, 5, ..., 10$. When m is even, or when m is odd and $0 \leqslant 2n \leqslant m - 3$, we find that as n increases, the index of stability will be non-increasing; thus proving that (17) can be used stably in the forward direction (for more details, see [14]). Actually, the above results have only been proven for the special case when n is an even integer; however, we could have used similar techniques to prove that forward recurrence using (17) is also stable when n is an odd integer.

Next we will handle the case when m is an odd integer and $2n \geqslant m + 1$. For this case, the forward recurrence will start at $2n = m + 1$; therefore, it will be easier to directly compute the index of stability (74). This time we can use (73), (74), and (78) to show that

$$\alpha^{(1)}(m+1, 2n) \sim \left(\frac{2}{Lr}\right)^m J_m(Lr)\, \Gamma(m+1);$$
$$m \text{ odd},\ 2n \geqslant m + 1,\ 2n \gg Lr. \tag{81}$$

Likewise, when $2n \ll Lr$ we can use (73), (74), and (76) to show that

$$\alpha^{(1)}(m+1, 2n) \sim \sqrt{\frac{\pi Lr}{2}}\, J_m(Lr) \times \frac{\Gamma(n + (1-m)/2)\, \Gamma(m+1)}{\Gamma(n + (1+m)/2)\, |\sin(Lr - n\pi - \pi/4)|};$$
$$m \text{ odd},\ 2n \geqslant m + 1,\ 2n \ll Lr. \tag{82}$$

Once again we find that the index of stability (see (81) or (82)) will be non-increasing as n increases. Therefore, (17) can be used stably in the forward direction for all values of m and Lr.

Finally, we will use the techniques in [14] to handle the stability analysis for (20). In order to simplify the analysis, we will combine the two recurrence relations in (20), thereby obtaining

$$j_{2m,0}(L, r) + d^{(2)}_{2m,0}\, j_{2(m+1),0}(L, r) = f^{(2)}_{2m,0}, \tag{83}$$

where

$$d^{(2)}_{2m,0} = \frac{(2m+1)^2}{r^2}$$
$$f^{(2)}_{2m,0} = \left\{ \frac{(2m+1)\, J_0(Lr)}{Lr} - J_1(Lr) \right\} \Big/ (rL^{2m}). \tag{84}$$

This time we find that the homogeneous solution is given by (see [13, (6.1.12)])

$$j_{2m,0}^{(2h)}(r) = \prod_{k=0}^{m-1} [-d_{2k,0}^{(2)}]^{-1} = \left(\frac{-r^2}{4}\right)^m \left[\frac{\Gamma(1/2)}{\Gamma(m+1/2)}\right]^2, \tag{85}$$

and that

$$\rho_{2m,0}^{(2)} = \left|\frac{j_{0,0}(L,r)\, j_{2m,0}^{(2h)}(r)}{j_{2m,0}(L,r)}\right| \sim \left(\frac{Lr}{2}\right)^{2m} \left[\frac{\Gamma(1/2)}{\Gamma(m+1/2)}\right]^2, \tag{86}$$

where we have made use of (76).

For even values of m, the recurrence relations in (20) will have the same kind of behavior as that of (83). We could have also performed a similar analysis in order to determine the behavior of (20) for odd values of m. Actually, for general values of m, it can be shown that the index of stability for (83) can be written as (see [14, (2.17)])

$$\alpha^{(2)}(k,m) = \rho_{m,0}^{(2)} / \rho_{k,0}^{(2)}, \tag{87}$$

where

$$\rho_{m,0}^{(2)} = \left|\frac{j_{0,0}(L,r)\, j_{m,0}^{(2h)}(r)}{j_{m,0}(L,r)}\right| \sim \left(\frac{Lr}{2}\right)^{m} \left[\frac{\Gamma(1/2)}{\Gamma((m+1)/2)}\right]^2. \tag{88}$$

Using [7, (106)], we find that $\rho_{m,0}$ reaches a maximum value when $m = m_{\max}$, where

$$m_{\max} = \operatorname{int}(Lr - 1). \tag{89}$$

Therefore, if we use $j_{m_{\max},0}(L,r)$ as a starting function, then reference to (87) and (88) shows that (83) can be used stably in the forward direction to compute $j_{m,0}(L,r)$ for $m = m_{\max}+1,\ m_{\max}+2, \ldots, 10$, and backward recurrence can be used stably for $m = m_{\max}-1,\ m_{\max}-2, \ldots, 3$.

APPENDIX C: EXPANSIONS FOR THE STARTING FUNCTIONS

We will use two different techniques to compute the starting functions $j_{m_{\max},0}(L,r)$ and $j_{m_{\max},1}(L,r)$. When $Lr > \mathrm{SD} + 4$, where SD is the number of desired significant digits, we can use [13, (9.2.5)] to write

$$\begin{aligned} j_{m,n}(L,r) \sim{}& r^{m-1}\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \left(-\frac{1}{4}\right)^k \\ &\times \left\{(n,2k)\left\{\cos\left[\frac{\pi}{4}(2n+1)\right]\int_{Lr}^{\infty} \frac{\cos t}{t^{2k+m+1/2}}\,dt \right.\right. \\ &+ \left.\sin\left[\frac{\pi}{4}(2n+1)\right]\int_{Lr}^{\infty} \frac{\sin t}{t^{2k+m+1/2}}\,dt\right\} \\ &- \frac{(n,2k+1)}{2}\left\{\cos\left[\frac{\pi}{4}(2n+1)\right]\right. \\ &\times \int_{Lr}^{\infty} \frac{\sin t}{t^{2k+m+3/2}}\,dt - \sin\left[\frac{\pi}{4}(2n+1)\right] \\ &\times \left.\left.\int_{Lr}^{\infty} \frac{\cos t}{t^{2k+m+3/2}}\,dt\right\}\right\}. \end{aligned} \tag{90}$$

Furthermore, the integrals in the above equation can be written in terms of incomplete gamma functions [13, (6.5.3)]

$$\begin{aligned} \int_{Lr}^{\infty} \frac{\cos t}{t^{\mu}}\,dt &= \Re[j^{\mu-1}\Gamma(1-\mu, jLr)] \\ \int_{Lr}^{\infty} \frac{\sin t}{t^{\mu}}\,dt &= -\Im[j^{\mu-1}\Gamma(1-\mu, jLr)]. \end{aligned} \tag{91}$$

Therefore, if we can find a way to compute the required sequence of incomplete gamma functions, then we can use (90) to compute the starting functions, $j_{m_{\max},0}(L,r)$ and $j_{m_{\max},1}(L,r)$.

Incomplete gamma functions also satisfy a first-order, non-homogeneous recurrence relation [13, (6.5.3), (6.5.22)],

$$\Gamma(-n+\tfrac{1}{2}, z) + d_n^{(3)}\Gamma(-(n+1)+\tfrac{1}{2}, z) = f_n^{(3)}, \tag{92}$$

where

$$\begin{aligned} d_n^{(3)} &= n + \tfrac{1}{2} \\ f_n^{(3)} &= z^{-n-1/2}e^{-z}. \end{aligned} \tag{93}$$

Applying the techniques for stability analysis in [14; 13, (6.1.12)], we find that the homogeneous solution for (92) is given by

$$\Gamma^{(h)}\left(-n+\frac{1}{2}\right) = \prod_{k=0}^{n-1} [-d_k^{(3)}]^{-1} = (-1)^n \frac{\Gamma(1/2)}{\Gamma(n+1/2)}. \tag{94}$$

Also, when $z \in S_\pi$, where

$$S_\pi = [z : |\angle z| < \pi], \tag{95}$$

it can be shown that

$$\Gamma\left(-n+\frac{1}{2}, z\right)=\int_z^{\infty} \frac{e^{-t}}{t^{n+1/2}}\, dt \sim z^{-n-1/2} e^{-z}; \qquad |z| \gg 1, n \geqslant 0. \quad (96)$$

Therefore, we find that the stability index is given by (see [14, (2.17)])

$$\alpha^{(3)}(k, n)=\rho_n^{(3)} / \rho_k^{(3)}, \quad (97)$$

where

$$\rho_n^{(3)}=\left|\frac{\Gamma(1/2, z)\, \Gamma^{(h)}(-n+1/2)}{\Gamma(-n+1/2, z)}\right| \sim\left|\frac{z^n \Gamma(1/2)}{\Gamma(n+1/2)}\right|; \qquad |z| \gg 1. \quad (98)$$

Once again, we can use [7, (106)] to show that ρ_n reaches a maximum value when $n=n_{\max}=\operatorname{int}(|z|-\frac{1}{2})$. Therefore, once we obtain the starting function $\Gamma(-n_{\max}+\frac{1}{2}, z)$, we can use (92) in the backward direction to obtain $\Gamma(-n+\frac{1}{2}, z)$ for $n=n_{\max}-1, ..., m_{\max}$, and we can use forward recurrence for $n=n_{\max}+1$, $n_{\max}+2, ...$ The continued fraction expansion [13, (6.5.31)] provides the most efficient method for obtaining the required starting function $\Gamma(-n_{\max}+\frac{1}{2}, z)$.

In summary, when $Lr > \mathrm{SD}+4$, we can use [13, (6.5.31)] to compute the starting function $\Gamma(-n_{\max}+\frac{1}{2}, jLr)$. Then we can use (90) to compute the desired integrals, $j_{m_{\max},0}(L, r)$ and $j_{m_{\max},1}(L, r)$, where the recurrence relation (92) is used to obtain the incomplete gamma functions in the asymptotic expansion.

When $Lr < \mathrm{SD}+4$, a different method is required for computing the starting functions $j_{m_{\max},0}(L, r)$ and $j_{m_{\max},1}(L, r)$. For this case, we find that (see (89)) $m_{\max} \leqslant \operatorname{int}(\mathrm{SD}+3)$. Therefore, if we compute the starting functions $j_{3,0}(L, r)$ and $j_{3,1}(L, r)$, then (20) can be used relatively stably in the forward direction for $\mathrm{SD} \leqslant 5$. It is true that we may loose some accuracy when we compute $j_{m,0}(L, r)$ and $j_{m,1}(L, r)$ for $m=4, 5, ..., 10$, but the loss will not be significant.

In order to compute these starting functions, we first split the integral into two pieces:

$$j_{3,n}(L, r)=\int_L^{9/r} \frac{J_n(r\lambda)}{\lambda^3}\, d\lambda+r^2 j_{3,n}(9, 1). \quad (99)$$

The second integral on the right-hand side of (99) can now be computed using (90) when $\mathrm{SD} \leqslant 5$. This integral only needs to be computed once since it is independent of r. In fact, we find that

$$\begin{aligned} j_{3,0}(9, 1) &= -3.320658 \times 10^{-4} \\ j_{3,1}(9, 1) &= -1.795596 \times 10^{-5}. \end{aligned} \quad (100)$$

Now, if we expand the Bessel function, in the other piece, in a power series [13, (9.1.10)], and integrate term-by-term, we find that

$$\begin{aligned} j_{3,0}(L, r) &= r^2 \left\{ j_{3,0}(9, 1)+\frac{1}{2}\left[\frac{1}{(Lr)^2}-\frac{1}{81}\right] + \frac{\ln(Lr/9)}{4}+\sum_{k=2}^{\infty} \frac{[9^{2k-2}-(Lr)^{2k-2}]}{(-4)^k [k!]^2 (2k-2)} \right\} \\ j_{3,1}(L, r) &= r^2 \left\{ j_{3,1}(9, 1) + \sum_{k=0}^{\infty} \frac{[9^{2k-1}-(Lr)^{2k-1}]}{(-4)^k k!(k+1)!\,(4k-2)} \right\}. \end{aligned} \quad (101)$$

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